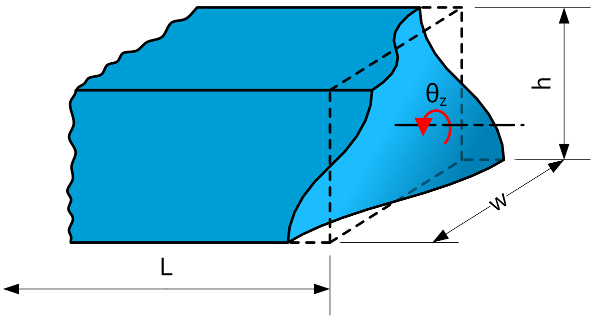
TORSION OF LEAF SPRINGS: RESTRAINED WARPING

Introduction

On page Beam theory: Torsion, the stiffness and shear stress due to applied torsion is presented. These equations assume the cross section of the beam is free to warp, however, this is not always true, specifically in leaf spring design.

Warping due to torsion

Torsion would cause the cross section of the beam to rotate. For non-circular cross sections, the cross section would warp, see figure. E.g., the corners of a rectangle would move out of the cross-sectional plane.



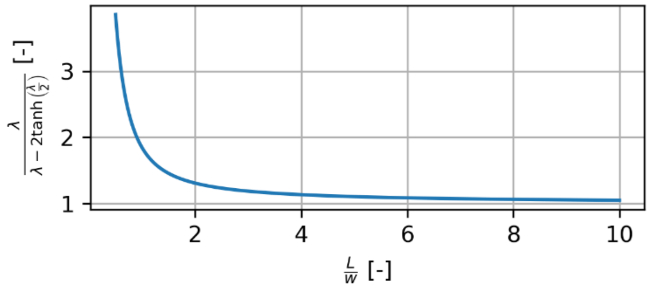
Restrained warping

When a cross section of a beam is constrained such that, the warping is restrained (e.g., leaf springs between stiff components), bending stresses will be introduced. Consequently, the torsion stiffness increases. Specifically for short and wide beams, the increase can be significant.

Torsion stiffness with restrained warping

The stiffness increase by restrained warp on both ends of the beam is given by:

$$K_z = \frac{GJ_T}{L}\left(\frac{\lambda}{\lambda - 2\tanh\left(\frac{\lambda}{2}\right)}\right)$$



where the dimensionless warping parameter is:

$$\lambda = L\sqrt{\frac{GJ_T}{EC_w}}$$

and J_T denotes the torsion and C_w the warping constant. A lower value of λ indicates a higher stiffness increase, see the graph which indicates the stiffness increase due to restrained warping.

Stresses with restrained warping

A beam with applied torsion will result in shear stresses. The maximum shear stress in the beam is given by:

$$\tau_{max} = \frac{\pi}{2}\frac{Gh}{L}\theta_{z,\,max} \approx \frac{3T}{wh^2}$$

Unlike beams with warping, the beams with restrained warping will also be submitted to bending stresses to counteract the warping. An estimation of the bending stresses is given by:

$$\sigma_{b,max} = \frac{4}{3}\frac{Ewh}{L^2}\theta_{z,\,max}$$

The maximum shear stress and maximum bending stress is not at the same place in the leaf spring. Thus, the maximum Von Mises stress can be roughly estimated with:

$$\sigma_{M,max} = \max\{\sqrt{3}\tau_{max}\ ;\ \sigma_{1,max}\}$$

where

$$\sigma_{1,max} = \sigma_{b,max}\sqrt{1 - 2\nu + \nu^2}$$

Leaf spring	Warping	One end restrained warping	Both ends restrained warping
Torsion stiffness	$K_z = \frac{GJ_T}{L}$	$K_z = \frac{GJ_T}{L}\left(\frac{\lambda}{\lambda - \tanh(\lambda)}\right) \approx \frac{GJ_T}{L}\left(1 + 0.16\frac{w^2}{L^2}\right)$ *	$K_z = \frac{GJ_T}{L}\left(\frac{\lambda}{\lambda - 2\tanh(\lambda/2)}\right) \approx \frac{GJ_T}{L}\left(1 + 0.65\frac{w^2}{L^2}\right)$ *
Torsion and warping constants for a rectangular cross section ($w > h$)	$J_T \approx \frac{1}{3}wh^3$	$C_w \approx \frac{1}{144}w^3h^3$	
Warping parameter	-	$\lambda = L\sqrt{\frac{GJ_T}{EC_w}} \approx \frac{L}{w}\sqrt{\frac{24}{1+\nu}}$ *	
Maximum shear stress	$\tau_{max} = \frac{Gh}{L}\theta_{z,\,max}$	$\tau_{max} = \frac{\pi}{2}\frac{Gh}{L}\theta_{z,\,max}$	
Bending stress	-	$\sigma_{b,max} = \frac{4}{3}\frac{Ewh}{L^2}\theta_{z,\,max}$	
Von Mises stress (*estimate*)	$\sqrt{3}\tau_{max}$	$\max\{\sqrt{3}\tau_{max}\ ;\ \sigma_{b,max}\sqrt{1 - 2\nu + \nu^2} \approx 0.7\sigma_{b,max}$ * $\}$	

* For most metals: $\nu = 0.3$

Sources:
- Nijenhuis, M. et al. (2022) *High torsion stiffness leafspring flexure element using distributed warping constraints.* Euspen's 22nd International Conference & Exhibition, Geneva
- Van den Berg, S. (2020) *Design Concepts and Strategies for Precision Engineering*. Berg Precision Publishing.