Introduction

In many mechatronic applications where a movement from A to B needs to be performed, a third order point to point motion profile is used.

To enable early insight in the relevant parameters of a motion profile it is useful to calculate and visualize the relevant parameters (position, speed, acceleration and jerk).

This sheet provides analytical formulas to calculate the quickest motion between point A to point B based on given maximum levels of speed, acceleration and jerk.
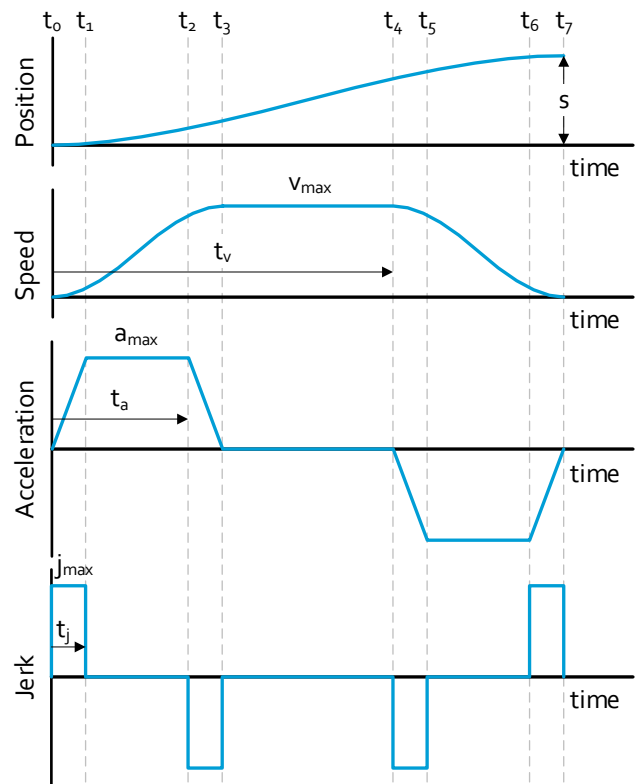
Approach

The typical time plot of the parameters of a third order profile is depicted on the right.

The difficulty in defining the motion trajectory is that the shape is not always the same. For example, there are cases where the maximum speed or acceleration level is not achieved, because there is not enough time to build up to the maximum before slowing down again. There are 6 different trajectory shapes that influence the motion profile.

The approach to find your motion profile is as follows:

1. Define which trajectory shape you have (I … VI) by comparing your required distance s and maximum velocity v_{max} with s_v, s_a and v_a (which are based on J_{max}, a_{max} and v_{max}).
2. Calculate the characteristic time intervals t_j, t_a, t_v based on your trajectory shape (I … VI).

3. Translate the characteristic time intervals to the $t_1 \ldots t_7$ values.
4. Using the formula's in the Motion parameters table the curves for jerk, acceleration, velocity and position on each time interval can be calculated.

1.	Trajectory shape						
	I	II	III	IV	V	VI	
v_{max}	<	>	<	<	>	>	$v_a = \dfrac{a_{max}^2}{j_{max}}$
s	>	<	<	<	>	>	$s_a = \dfrac{2 \cdot a_{max}^3}{j_{max}^2}$
			>	<	>	<	$s_v = v_{max}\left[M\left(2\sqrt{\dfrac{v_{max}}{j_{max}}}\right) + N\left(\dfrac{v_{max}}{a_{max}} + \dfrac{a_{max}}{j_{max}}\right)\right]$ $M = 1, N = 0 \ if \ v_{max}j_{max} < a_{max}^2$ $M = 0, N = 1 \ if \ v_{max}j_{max} \geq a_{max}^2$

3.	Time intervals as function of characteristic time intervals	
$t_1 = t_j$	$t_5 = t_v + t_j$	
$t_2 = t_a$	$t_6 = t_v + t_a$	
$t_3 = t_a + t_j$	$t_7 = t_v + t_j + t_a$	
$t_4 = t_v$		

2.	Characteristic time intervals					
Trajectory shape	I	II	III	IV	V	VI
t_j	$\sqrt{\dfrac{v_{max}}{j_{max}}}$	$\sqrt[3]{\dfrac{s}{2 \cdot j_{max}}}$	$\sqrt{\dfrac{v_{max}}{j_{max}}}$	$\sqrt[3]{\dfrac{s}{2 \cdot j_{max}}}$	$\dfrac{a_{max}}{j_{max}}$	$\dfrac{a_{max}}{j_{max}}$
t_a	t_j	t_j	t_j	t_j	$\dfrac{v_{max}}{a_{max}}$	$\dfrac{1}{2}\left(\sqrt{\dfrac{4 \cdot s \cdot j_{max}^2 + a_{max}^3}{a_{max} \cdot j_{max}^2}} - \dfrac{a_{max}}{j_{max}}\right)$
t_v	$\dfrac{s}{v_{max}}$	$2 \cdot t_j$	$\dfrac{s}{v_{max}}$	$2 \cdot t_j$	$\dfrac{s}{v_{max}}$	$t_a + t_j$

4.	Motion parameters			
	Jerk	Acceleration	Velocity	Position
t0…t1	j_{max}	$j_{max} \cdot (t - t_0)$	$\frac{1}{2} \cdot j_{max} \cdot (t - t_0)^2$	$\frac{1}{6} \cdot j_{max} \cdot (t - t_0)^3$
t1…t2	0	$a_1 = a_2$	$v_1 + a_1 \cdot (t - t_1)$	$p_1 + v_1 \cdot (t - t_1) + \frac{1}{2} \cdot a_1 \cdot (t - t_1)^2$
t2…t3	$-j_{max}$	$a_2 - j_{max} \cdot (t - t_2)$	$v_2 + a_2 \cdot (t - t_2) + \frac{1}{2} \cdot -j_{max} \cdot (t - t_2)^2$	$p_2 + v_2 \cdot (t - t_2) + \frac{1}{2} \cdot a_2 \cdot (t - t_2)^2 + \frac{1}{6} \cdot -j_{max} \cdot (t - t_2)^3$
t3…t4	0	0	$v_3 = v_4$	$p_3 + v_3 \cdot (t - t_3)$
t4…t5	$-j_{max}$	$-j_{max} \cdot (t - t_4)$	$v_4 + \frac{1}{2} \cdot -j_{max} \cdot (t - t_4)^2$	$p_4 + v_4 \cdot (t - t_4) + \frac{1}{6} \cdot -j_{max} \cdot (t - t_4)^3$
t5…t6	0	$a_5 = a_6$	$v_5 + a_5 \cdot (t - t_5)$	$p_5 + v_5 \cdot (t - t_5) + \frac{1}{2} \cdot a_5 \cdot (t - t_5)^2$
t6…t7	j_{max}	$a_6 + j_{max} \cdot (t - t_6)$	$v_6 + a_6 \cdot (t - t_6) + \frac{1}{2} \cdot j_{max} \cdot (t - t_6)^2$	$p_6 + v_6 \cdot (t - t_6) + \frac{1}{2} \cdot a_6 \cdot (t - t_6)^2 + \frac{1}{6} \cdot j_{max} \cdot (t - t_6)^3$

Sources:
• Third-order trajectory planning for high accuracy point-to-point motion - Haihua MU et al.

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