Introduction

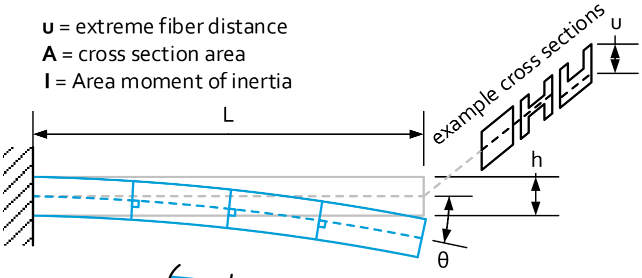
Beam theory is such a common engineering fundamental; it is impossible to be omitted from almost any engineering-specialism. However, this sheet incorporates stress and stiffness as well. For more information on, and calculations of the area moment of inertia *I*, see sheet: *Area moment of inertia*.

Validity

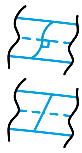
Euler-Bernoulli beam theory is only valid with the following assumptions:

- Cross sections of the beam do not deform in a significant manner under the application of transverse or axial loads and can be assumed as rigid.
- During deformation, the cross section of the beam is assumed to remain planar and normal to the deformed axis of the beam.

Generally, these criteria are met when the beam is a slender beam with small rotations. This means the slenderness of the beam (ratio L/h) should be larger than 10 and the rotation of the neutral axis (θ) should be smaller than 5°.

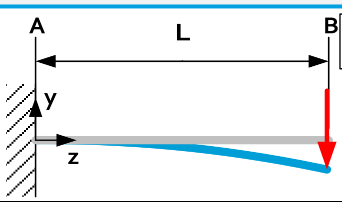
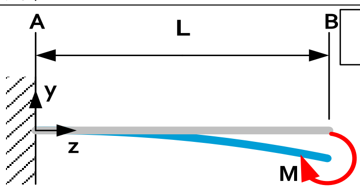
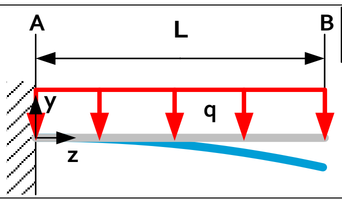
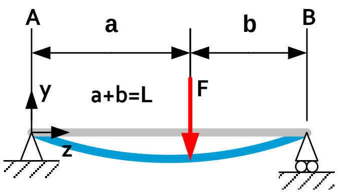


Not valid for:



Cross section not planar

Cross section not normal to the deformed axis

Load case	Curvature θ Sag δy	Reaction force R Shear force D Reaction moment M_R	Stress σ Stiffness C or K
1 	$\theta_A = 0$ $\theta_B = \frac{FL^2}{2EI_x}$ $\delta y_z = -\frac{(Fz^2)(3L-z)}{6EI_x}$ $\delta y_{max} = -\frac{FL^3}{3EI_x} @ z = L$	$R_A = F$ $R_B = N.A.$ $D_z = F$ $M_{Rz} = F(z-L)$ $M_{Rmax} = -FL @ z = 0$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x}u = -\frac{\lvert F\rvert u(z-L)}{I_x}$ $\lvert\sigma_{max}\rvert = \frac{\lvert F\rvert Lu}{I_x} @ z = 0$ $\lvert C_y\rvert = \left\lvert\frac{F}{\delta y_{z(F)}}\right\rvert = \frac{3EI_x}{L^3} @ z = L$
2 	$\theta_A = 0$ $\theta_B = \frac{ML}{EI_x}$ $\delta y_z = -\frac{Mz^2}{2EI_x}$ $\delta y_{max} = -\frac{ML^2}{2EI_x} @ z = L$	$R_A = 0$ $R_B = N.A.$ $D_z = N.A.$ $M_{Rz} = M$ $M_{Rmax} = M @ z = const.$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x}u = \frac{\lvert M\rvert u}{I_x}$ $\lvert\sigma_{max}\rvert = \frac{\lvert M\rvert u}{I_x} @ z = const.$ $\lvert K_y\rvert = \left\lvert\frac{M}{\theta_B}\right\rvert = \frac{EI_x}{L} @ z = L$
3 	$\theta_A = 0$ $\theta_B = \frac{qL^3}{6EI_x}$ $\delta y_z = -\frac{qz^2}{(24EI_x)(6L^2-4Lz+z^2)}$ $\delta y_{max} = -\frac{qL^4}{8EI_x} @ z = L$	$R_A = qL$ $R_B = N.A.$ $D_z = q(L-z)$ $M_{Rz} = -\frac{q(L-z)^2}{2}$ $M_{Rmax} = -\frac{qL^2}{2} @ z = 0$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x}u = \frac{\lvert q\rvert u(L-z)^2}{2I_x}$ $\lvert\sigma_{max}\rvert = \frac{\lvert q\rvert L^2 u}{2I_x} @ z = 0$ $\lvert C_y\rvert = \left\lvert\frac{q}{\delta y_{max}}\right\rvert = \frac{8EI_x}{L^4} @ z = L$
4 	$0 \le z \le a$ $\theta_A = \frac{Fab(L+b)}{6EI_xL}$ $\theta_B = \frac{Fab(L+a)}{6EI_xL}$ $\delta y_z = -\frac{Fab^2}{6EI_x}\left[\left(1+\frac{L}{b}\right)\frac{z}{L} - \frac{z^3}{abL}\right]$ $\delta y_{max} = -\frac{Fb\sqrt{(L^2-b^2)^3}}{9\sqrt{3}EI_xL}$ $@ z = \sqrt{\frac{L^2-b^2}{3}}$ only if $a > b$	$R_A = \frac{Fb}{L}$ $R_B = \frac{Fa}{L}$ $D_z = \frac{-Fb}{L}$ $M_{Rz} = \frac{Fbz}{a+b}$ $M_{Rmax} = \frac{Fba}{a+b} @ z = a$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x}u = \frac{\lvert F\rvert bzu}{I_xL}$ $\lvert\sigma_{max}\rvert = \frac{\lvert F\rvert bau}{I_x(a+b)} @ z = a$ $\lvert C_y\rvert = \left\lvert\frac{F}{\delta y_{z(F)}}\right\rvert = \frac{3EI_xL}{a^2b^2} @ z = a$
	$a \le z \le L$ $\theta_A = \frac{Fab(L+b)}{6EI_xL}$ $\theta_B = \frac{Fab(L+a)}{6EI_xL}$ $\delta y_z = -\frac{Fa^2b}{6EI_x}\left[\left(1+\frac{L}{a}\right)\frac{L-z}{L} - \frac{(L-z)^3}{abL}\right]$ $\delta y_{max} = -\frac{Fa\sqrt{(L^2-a^2)^3}}{9\sqrt{3}EI_xL}$ $@ z = L - \sqrt{\frac{L^2-a^2}{3}}$ only if $a < b$	$R_A = \frac{Fb}{L}$ $R_B = \frac{Fa}{L}$ $D_z = \frac{Fb}{L} - F$ $M_{Rz} = \left(\frac{Fbz}{L}\right) - F(z-a)$ $M_{Rmax} = \frac{Fba}{L} @ z = a$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x}u = \frac{\left\lvert\left(\frac{Fbz}{L}\right)-F(z-a)\right\rvert}{I_x}u$ $\lvert\sigma_{max}\rvert = \frac{\lvert F\rvert bau}{I_xL} @ z = a$ $\lvert C_y\rvert = \left\lvert\frac{F}{\delta y_{z(F)}}\right\rvert = \frac{3EI_xL}{a^2b^2} @ z = a$

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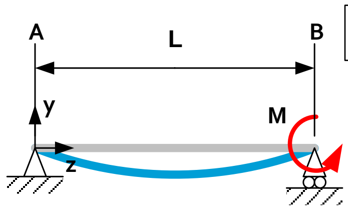
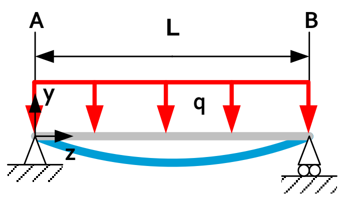
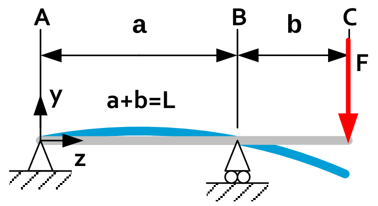
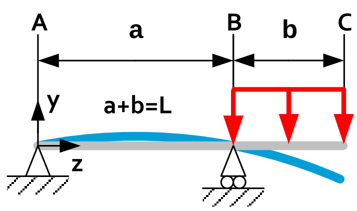
Load case		Curvature θ sag δy	Reaction force R Shear force D Reaction moment M_R	Stress σ Stiffness C
5 		$\theta_A = \frac{ML}{6EI_x}$ $\theta_B = \frac{ML}{3EI_x}$ $\delta y_z = -\frac{ML^2}{6EI_x}\left[\frac{z}{L} - \left(\frac{z}{L}\right)^3\right]$ $\delta y_{max} = -\frac{ML^2}{9\sqrt{3}EI_x}\ @\ z = \frac{L}{\sqrt{3}}$	$R_A = \frac{M}{L}$ $R_B = -\frac{M}{L}$ $D_z = \frac{M}{L}$ $M_{Rz} = \left(\frac{M}{L}\right) z$ $M_{Rmax} = M\ @\ z = L$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x} u = \frac{\lvert M\rvert zu}{LI_x}$ $\lvert\sigma_{max}\rvert = \frac{\lvert M\rvert u}{I_x}\ @\ z = L$ $\lvert K_y\rvert = \left\lvert\frac{M}{\theta_B}\right\rvert = \frac{3EI_x}{L}\ @\ z = L$
6 		$\theta_A = \frac{qL^3}{24EI_x}$ $\theta_B = \frac{qL^3}{24EI_x}$ $\delta y_z = -\frac{qL^4}{24EI_x}\left[\frac{z}{L} - 2\left(\frac{z}{L}\right)^3 + \left(\frac{z}{L}\right)^4\right]$ $\delta y_{max} = -\frac{5}{384}\frac{qL^4}{EI_x}\ @\ \frac{L}{2}$	$R_A = \frac{qL}{2}$ $R_B = \frac{qL}{2}$ $D_z = \frac{qL}{2} - qz$ $M_{Rz} = \frac{qz(L-z)}{2}$ $M_{Rmax} = \frac{qL^2}{8}\ @\ z = \frac{L}{2}$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x} u = \frac{\lvert qz(L-z)u\rvert}{2I_x}$ $\lvert\sigma_{max}\rvert = \frac{\lvert q\rvert L^2 u}{8I_x}\ @\ z = \frac{L}{2}$ $\lvert C_y\rvert = \left\lvert\frac{q}{\delta y_{max}}\right\rvert =$ $\frac{5\lvert q\rvert L^4}{384EI_x}\ @\ z = \frac{L}{2}$
7 	$0 \le z \le a$	$\theta_A = \frac{FbL}{6EI_x}$ $\theta_B = \frac{FbL}{3EI_x}$ $\delta y_z = \frac{Fba^2}{6EI_x}\left[\frac{z}{a} - \left(\frac{z}{a}\right)^3\right]$ $\delta y_{max} = \frac{Fba^2}{9\sqrt{3}EI_x}\ @\ z = \frac{a}{\sqrt{3}}$	$R_A = -\frac{Fb}{a}$ $R_B = F + \frac{Fb}{a}$ $D_z = -\left(\frac{Fb}{a}\right)$ $M_{Rz} = -\frac{Fbz}{a}$ $M_{Rmax} = -Fb\ @\ z = a$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x} u = \frac{\lvert F\rvert bzu}{aI_x}$ $\lvert\sigma_{max}\rvert = \frac{\lvert F\rvert bu}{I_x}\ @\ z = a$ $\lvert C_y\rvert = \left\lvert\frac{F}{\delta y_{z(F)}}\right\rvert = \frac{3EI_x}{b^2 L}\ @\ z = L$
	$a \le z \le L$	$\theta_C = \frac{Fb(2a+3b)}{3EI_x}$ $\delta y_z = -\frac{F((-z)+a)}{6EI_x}[ab - 3bz + z^2 -$ $2az + a^2]$ $\delta y_{max} = -\frac{Fb^2L}{3EI_x}\ @\ z = L$	$R_A = -\frac{Fb}{a}$ $R_B = F + \frac{Fb}{a}$ $D_z = F$ $M_{Rz} = -F[(-z) + a + b]$ $M_{Rmax} = -Fb\ @\ z = a$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x} u = \frac{\lvert F\rvert[(-z)+L)}{I_x} u$ $\lvert\sigma_{max}\rvert = \frac{\lvert F\rvert bu}{I_x}\ @\ z = a$ $\lvert C_y\rvert = \left\lvert\frac{F}{\delta y_{z(F)}}\right\rvert = \frac{3EI_x}{b^2 L}\ @\ z = L$
8 	$0 \le z \le a$	$\theta_A = \frac{qb^2a}{12EI_x}$ $\theta_B = \frac{qb^2a}{6EI_x}$ $\delta y_z = \frac{qb^2a^2}{12EI_x}\left[\frac{z}{a} - \left(\frac{z}{a}\right)^3\right]$ $\delta y_{max} = \frac{qb^2a^2}{18\sqrt{3}EI_x}\ @\ z = \frac{a}{\sqrt{3}}$	$R_A = -\frac{qb^2}{2a}$ $R_B = \frac{qb(b+2a)}{2a}$ $D_z = -\frac{qb^2}{2a}$ $M_{Rz} = -\left(\frac{qb^2}{2a}\right) z$ $M_{Rmax} = -\frac{qb^2}{2}\ @\ z = a$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x} u = \frac{\lvert q\rvert b^2 zu}{2aI_x}$ $\lvert\sigma_{max}\rvert = \frac{\lvert q\rvert b^2 u}{2I_x}\ @\ z = a$ $\lvert C_y\rvert = \left\lvert\frac{q}{\delta y_{max}}\right\rvert = \frac{18\sqrt{3}EI_x}{b^2a^2}\ @\ z = \frac{a}{\sqrt{3}}$
	$a \le z \le L$	$\theta_C = \frac{qb^2L}{6EI_x}$ $\delta y_z = -\frac{qb^4}{24EI_x}\left[4\frac{a}{b}\frac{z-a}{b} + 6\left(\frac{z-a}{b}\right)^2 -\right.$ $\left. 4\left(\frac{z-a}{b}\right)^3 + \left(\frac{z-a}{b}\right)^4\right]$ $\delta y_{max} = -\frac{qb^3(4a+3b)}{24EI_x}\ @\ z = L$	$R_A = -\frac{qb^2}{2a}$ $R_B = \frac{qb(b+2a)}{2a}$ $D_z = qb - q(z+a)$ $M_{Rz} =$ $\frac{-q(b^2-2bz+2ba+z^2-2za+a^2)}{2}$ $M_{Rmax} = -\frac{qb^2}{2}\ @\ z = a$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x} u =$ $\frac{\lvert qu(b^2-2bz+2ba+z^2-2za+a^2)\rvert}{2I_x}$ $\lvert\sigma_{max}\rvert = \frac{\lvert q\rvert b^2 u}{2I_x}\ @\ z = a$ $\lvert C_y\rvert = \left\lvert\frac{q}{\delta y_{max}}\right\rvert = \frac{24EI_x}{b^3(4a+3b)}\ @z = L$

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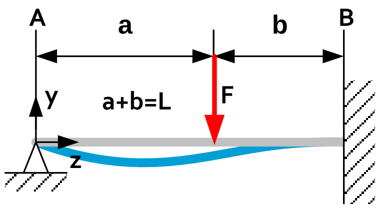
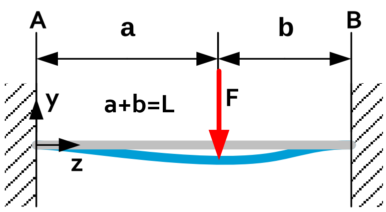
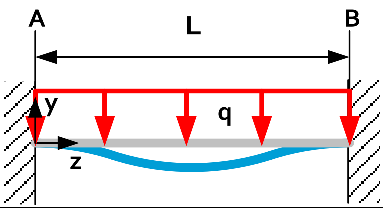
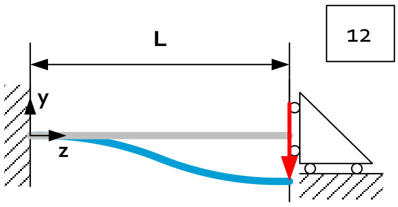
Load case		Curvature θ sag δy	Reaction force R Shear force D Moment M	Stress σ Stiffness C
9 	$0 \le z \le a$	$\theta_A = \frac{Fab^2}{4EI_xL}$ $\theta_B = 0$ $\delta y_z = -\frac{FLb^2}{4EI_x}\left[\frac{az}{L^2} - \frac{2}{3}\left(1+\frac{a}{2L}\right)\left(\frac{z}{L}\right)^3\right]$ $\delta y_{max} = -\frac{ab^2F\sqrt{\frac{a}{2L+a}}}{6EI_x} @z = L \cdot \sqrt{\frac{\frac{a}{2L}}{1+\frac{a}{2L}}}$ Only if a ≥ 0.414L	$R_A = F\left(\frac{b}{L}\right)^2\left(1+\frac{a}{2L}\right)$ $R_B = F\left(\frac{a}{L}\right)^2\left(1+\frac{b}{2L}+\frac{3}{2}\frac{b}{a}\right)$ $D_z = F\left(\frac{b}{L}\right)^2\left(1+\frac{a}{2L}\right)$ $M_{Rz} = Fz\left(\frac{b}{L}\right)^2\left(1+\frac{a}{2L}\right)$ $M_{Rmax} = \frac{Fab^2}{L^2}\left(1+\frac{a}{2L}\right) @ z =$ a, with $a \le 0.414L$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x}u = \frac{\lvert F\rvert zb^2u(2L+a)}{2L^3I_x}$ $\lvert\sigma_{max}\rvert = \frac{\lvert F\rvert ab^2u\left(1+\frac{a}{2L}\right)}{L^2I_x}$ $@z=a \quad a \le 0.414L$ $\lvert C_y\rvert = \left\lvert\frac{F}{\delta y_{z(F)}}\right\rvert = \left\lvert\frac{12EI_xL^3}{b^2a^2[3L^2-2aL-a^2]}\right\rvert @ z = a$
	$a \le z \le L$	$\theta_A = \frac{Fab^2}{4EI_xL}$ $\theta_B = 0$ $\delta y_z = \frac{Fa(L-z)^2}{12L^3}\frac{2La^2-3L^2z+a^2z}{EI_x}$ $\delta y_{max} = \delta y_z(z_{max})$ $@ z_{max} = L \cdot \left[\frac{2aL+bL-ba}{2aL+3bL+ba}\right]$ Only if a ≤ 0.414L	$R_A = F\left(\frac{b}{L}\right)^2\left(1+\frac{a}{2L}\right)$ $R_B = F\left(\frac{a}{L}\right)^2\left(1+\frac{b}{2L}+\frac{3}{2}\frac{b}{a}\right)$ $D_z = -F\left(\frac{a}{L}\right)^2\left(1+\frac{b}{2L}+\frac{3}{2}\frac{b}{a}\right)$ $M_{Rz} = Fz\left(\frac{b}{L}\right)^2\left(1+\frac{a}{2L}\right) - F(z-a)$ $M_{Rmax} = -F\frac{ab}{L}\left(1-\frac{b}{2L}\right)$ $@ z = L$, with $a \ge 0.414L$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x}u = \frac{\lvert Fu(2zLb^2+b^2za-2L^3z+2L^3a)\rvert}{2L^3I_x}$ $\lvert\sigma_{max}\rvert = \frac{\left\lvert Fabu\left(1-\frac{b}{2L}\right)\right\rvert}{LI_x}$ $@ z = L$, with $a \ge 0.414L$ $\lvert C_y\rvert = \left\lvert\frac{F}{\delta y_{z(F)}}\right\rvert = \left\lvert\frac{12EI_xL^3}{b^2a^2[3L^2-2aL-a^2]}\right\rvert @ z = a$
10 	$0 \le z \le a$	$\theta_A = 0$ $\theta_B = 0$ $\delta y_z = -\frac{FLb^2}{6EI_x}\left[3\frac{a}{L}\left(\frac{z}{L}\right)^2 - \left(1+\frac{2a}{L}\right)\left(\frac{z}{L}\right)^3\right]$	$R_A = F\left(\frac{b}{L}\right)^2\left(1+\frac{2a}{L}\right)$ $R_B = F\left(\frac{a}{L}\right)^2\left(1+\frac{2b}{L}\right)$ $D_z = F\left(\frac{b}{L}\right)^2\left(1+\frac{2a}{L}\right)$ $M_{Rz} = \frac{Fb^2(zL+2za+aL)}{L^3}$ $M_{Rmax} = -\frac{Fab^2}{L^2} @ z = 0$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x}u = \frac{\lvert F\rvert b^2u(zL+2za+aL)}{I_xL^3}$ $\lvert\sigma_{max}\rvert = \frac{\lvert F\rvert uab^2}{L^2I_x} @z = 0$ $\lvert C_y\rvert = \left\lvert\frac{F}{\delta y_{z(F)}}\right\rvert = \frac{3EI_xL^3}{b^2a^3(L-a)} @ z = a$
	$a < b$	$\delta y_{max} = -\frac{2}{3}\frac{Fa^2b^3}{EI_xL^2}\left(\frac{1}{1+\frac{2b}{L}}\right)^2$ $@z = L - L * \frac{1}{1+\frac{L}{2b}}$		
	$a \le z \le L$	$\theta_A = 0$ $\theta_B = 0$ $\delta y_z = -\frac{FLa^2}{6EI_x}\left[3\frac{b}{L^3}(L-z)^2 - \left(1+\frac{2b}{L}\right)\frac{(L-z)^3}{L^3}\right]$	$R_A = F\left(\frac{b}{L}\right)^2\left(1+\frac{2a}{L}\right)$ $R_B = F\left(\frac{a}{L}\right)^2\left(1+\frac{2b}{L}\right)$ $D_z = -F\left(\frac{a}{L}\right)^2\left(1+\frac{2b}{L}\right)$ $M_{Rz} = \frac{F[b^2zL+2b^2za+b^2aL-L^3z+L^3a]}{L^3}$ $M_{Rmax} = -\frac{Fba^2}{L^2} @ z = L$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x}u = \frac{\lvert Fu[b^2zL+2b^2za+b^2aL-L^3z+L^3a]\rvert}{L^3I_x}$ $\lvert\sigma_{max}\rvert = \frac{\lvert F\rvert uba^2}{L^2I_x} @ z = L$ $\lvert C_y\rvert = \left\lvert\frac{F}{\delta y_{z(F)}}\right\rvert = \frac{3EI_xL^3}{b^2a^3(L-a)} @ z = a$
	$a > b$	$\delta y_{max} = -\frac{2}{3}\frac{Fa^3b^2}{EI_xL^2}\left(\frac{1}{1+\frac{2a}{L}}\right)^2 @ z = L\frac{1}{1+\frac{L}{2a}}$		
11 		$\theta_A = 0$ $\theta_B = 0$ $\delta y_z = -\frac{qL^4}{24EI_x}\left[\left(\frac{z}{L}\right)^2 - 2\left(\frac{z}{L}\right)^3 + \left(\frac{z}{L}\right)^4\right]$ $\delta y_{max} = -\frac{qL^4}{384EI_x} @ z = \frac{L}{2}$	$R_A = \frac{1}{2}qL$ $R_B = \frac{1}{2}qL$ $D_z = \frac{qL}{2} - qz$ $M_{Rz} = \frac{q(-6z^2+6Lz-L^2)}{12}$ $M_{Rmax} = -\frac{qL^2}{12} @ \frac{z=0}{z=L}$	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x}u = \frac{qu\lvert(-6z^2+6Lz-L^2)\rvert}{12I_x}$ $\lvert\sigma_{max}\rvert = \frac{\lvert q\rvert L^2u}{12I_x} @ \frac{z=0}{z=L}$ $\lvert C_y\rvert = \left\lvert\frac{q}{\delta y_{max}}\right\rvert = \frac{384EI_x}{L^4} @ z = \frac{L}{2}$
12 		$\theta_A = 0$ $\theta_B = 0$ $\delta y_z = -\frac{Fz^2(3L-2z)}{12EI_x}$ $\delta y_{max} = -\frac{FL^3}{12EI_x} @ z = L$	$R_A = F$ $R_B = 0$ $D_z = F$ $M_z = \frac{FL-2Fz}{2}$ $M_{Rmax} = \frac{FL}{2}$ resp. $-\frac{FL}{2}$ $@ z = 0$ resp. L	$\lvert\sigma_z\rvert = \frac{\lvert M_{Rz}\rvert}{I_x}u = \frac{\lvert u(FL+2Fz)\rvert}{2I_x}$ $\lvert\sigma_{max}\rvert = \frac{\lvert F\rvert Lu}{2I_x} @ \frac{z=0}{z=L}$ $\lvert C_y\rvert = \left\lvert\frac{F}{\delta y_{max}}\right\rvert = \frac{12EI_x}{L^3} @ z = L$

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